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ESTIMATING A STANDARD DEVIATION WITH *U*-STATISTICS OF DEGREE MORE THAN TWO: THE NORMAL CASE

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Abstract

We consider unbiased estimation of σ in a $N(\mu, \sigma^2)$ population. Traditional unbiased estimators consist of appropriate multiples of both the sample standard deviation S, that is, $T^{(1)}$ and Gini's mean difference (GMD), that is, $T^{(2)}$. Both $T^{(1)}, T^{(2)}$ depend upon U-statistics associated with symmetric kernels of degree two. In this paper, we develop a new approach of constructing higher-order unbiased U-statistics $T^{(3)}, T^{(4)}$, and $T^{(5)}$ for σ based upon symmetric kernels with degree three, four, and four, respectively. From this investigation, we find that the new unbiased estimators $T^{(3)}, T^{(4)}$, and $T^{(5)}$ for σ; (i) go practically head-to-head with the existing estimators $T^{(1)}$ and $T^{(2)}$, (ii) $T^{(4)}$ beats $T^{(2)}$, and (iii) $T^{(3)}$ very nearly beats $T^{(2)}$, whether n is small or moderately large. In other words, it is our belief that this new approach appears very promising.

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1. Introduction

The notion of variation has traditionally belonged to the core of statistics. One measure of variation that clearly stands out is obviously the *sample variance* or a *standard deviation*, which estimates a population variance σ^2 or a standard deviation σ.

In this paper, we focus on unbiased estimation of the standard deviation σ in a normal population. Suppose that $X_1, \ldots, X_n$ form a random sample, that is, these are independent and identically distributed (i.i.d.) random variables following a $N(\mu, \sigma^2)$ distribution. We denote the sample mean, $\overline{X} = n^{-1}\sum_{i=1}^{n} X_i$, and the sample variance,

$$S^2 = (n-1)^{-1}\sum_{i=1}^{n} (X_i - \overline{X})^2, \text{ for } n \geq 2.$$

We suppose that $0 < \sigma < \infty$.

From the expression of S^2, it is clear that it compares each observation with the sample average. However, one can equivalently see an alternative expression of the sample variance as follows:

$$S^2 = \binom{n}{2}^{-1} \sum_{1 \leq i_1 < i_2 \leq n} \frac{1}{2}(X_{i_1} - X_{i_2})^2. \tag{1.1}$$

From (1.1), it becomes abundantly clear that every X_i is compared with every other X_j in defining S^2.

The representation in (1.1) is called a *U-statistic* that belongs to a class of unbiased estimators (of σ^2), which was introduced by Hoeffding [10]. The U-statistic (1.1) is associated with a symmetric kernel, $g(x_1, x_2) = \frac{1}{2}(x_1 - x_2)^2$. Hoeffding's U-statistic is generally defined in (2.1) that is based on a suitable symmetric kernel of degree $m(< n)$.

1.1. A brief literature review

The literature on U-statistics is vast as well as rich. A long journey of the theory of U-statistics began with the pioneering contributions of Hoeffding [10, 11]. Numerous books devoted to topics on nonparametric statistics include some material on U-statistics. We may cite, for example, Puri and Sen [20], Sen [21], Mukhopadhyay and Solanky [17], Jureckova and Sen [13], Ghosh et al. [6], Hettmansperger and McKean [9], and Hollander and Wolfe [12]. One will also find a number of books dedicated exclusively to theory and practice of U-statistics including Lee [16], Kowalski and Tu [15], and Korolyuk and Borovskich [14].

In order to estimate σ unbiasedly, the *minimum variance unbiased estimator* (MVUE) comes to our minds first. It is a suitable multiple of S, which is denoted by $T^{(1)}$ in (2.4). There is another widely applied unbiased estimator, $T^{(2)}$, of σ defined in (2.5), which is a suitable multiple of *Gini's mean difference* (GMD). GMD was originally developed by Gini [7, 8]. Nair [19] constitutes one of the early contributions, which discussed the role of GMD in estimation theory for a normal distribution as well as some selected non-normal distributions. Yitzhaki [24] had dealt with GMD in estimating a standard deviation in a non-normal distribution. Both Downton [5] and D'Agostino [4] worked with an ordered version of GMD and came up with estimates of σ in a normal distribution.

Since the pioneering work of Gini, many statistical scientists, economists, and others were also drawn into numerous interpretations and generalizations driven by GMD in various directions. We may cite a selection of recent contributions by Sen [22], Sen [23], and Arnold [2, 3] among others.

1.2. An outline of this paper

In this paper, we introduce a basic methodology for constructing unbiased estimators of σ by U-statistics with kernels of degree $m > 2$ in a normal distribution. Section 2 begins with some preliminaries, illustrations, and motivations behind our approach by constructing new higher-order unbiased U-statistics $T^{(3)}$, $T^{(4)}$, and $T^{(5)}$ for σ with degree three and four displayed by (2.7), (2.8), and (2.10), respectively. For brevity, we do not include illustrations with U-statistics of degree beyond four.

Both $T^{(1)}$ from (2.4) and $T^{(2)}$ from (2.5) clearly rely upon kernels of degree two, and they stand as two heavily studied unbiased estimators of σ. We have decided to compare $T^{(3)}$, $T^{(4)}$, $T^{(5)}$ with $T^{(1)}$ and $T^{(2)}$ with the focal point of comparison being $T^{(2)}$ because we do not expect to beat $T^{(1)}$, the MVUE for σ. We first compare performances of all five estimators by using simulations and these preliminary findings are summarized in Tables 1 and 2. It may not be a bad idea to compare $T^{(3)}$, $T^{(4)}$, and $T^{(5)}$ with other existing unbiased estimators of σ, but we refrain from doing so in order to keep the length of this introductory discourse on our new approach within reason.

The exact variances of $T^{(1)}$, $T^{(2)}$, and $T^{(3)}$ are then laid out in (3.1)-(3.3) of Section 3 followed by the comparisons of their efficiencies (Subsection 3.1). The exact variances of $T^{(4)}$ and $T^{(5)}$ are hard to evaluate. Hence, large-sample approximations of the variances of $T^{(1)}$ through $T^{(5)}$ are then summarized in Subsection 3.2. Their proofs are rather involved and hence we have provided some of the key steps of all derivations in Subsections 4.1-4.4. Some crucial intermediate steps are put together as Lemmas 4.1-4.6 in Subsection 4.5.

In summary, it is clear from our investigation that three new unbiased estimators $T^{(3)}$, $T^{(4)}$, and $T^{(5)}$ for σ; (i) go practically head-to-head with existing estimators $T^{(1)}$ and $T^{(2)}$, (ii) $T^{(4)}$ beats $T^{(2)}$, and (iii) $T^{(3)}$ very nearly beats $T^{(2)}$, whether n is small or moderately large.

2. Some Preliminaries and Illustrations

Hoeffding's [10] U-statistic corresponding to a functional $\theta \equiv \theta(F)$ is customarily written as

$$U \equiv U_n^{(m)} = \binom{n}{m}^{-1} \sum_{n,m} g^{(m)}(X_{i_1}, X_{i_2}, \ldots, X_{i_m}), \tag{2.1}$$

where $\sum_{n,m}$ denotes the summation over all possible combinations of indices $i_1, i_2, \ldots, i_m$ such that $1 \le i_1 < i_2 < \ldots < i_m \le n$. Here, $g^{(m)}(.)$ is a symmetric kernel of degree m such that $E_F[g^{(m)}(X_1, \ldots, X_m)] = \theta(F)$, $n > m$. We will use a more descriptive notation shortly to streamline a series of different unbiased estimators of σ that we propose.

Let $g_{1,1}^{(2,1)}$ denote the kernel $\frac{1}{2}(x_i - x_j)^2$ of degree 2 associated with S^2. In other words, we will streamline our notation by expressing

$$S^2 \equiv U_{n,1,1}^{(2,1)} = \binom{n}{2}^{-1} \sum_{1 \le i_1 < i_2 \le n} g_{1,1}^{(2,1)}(X_{i_1}, X_{i_2}), \tag{2.2}$$

where $U_{n,1,1}^{(2,1)}$ happens to be the U-statistic from (2.1) with $m = 2$. In general, we will denote a U-statistic $U_{n,k,l}^{(m,p)}$ corresponding to a kernel $g_{k,l}^{(m,p)}(x_1, \ldots, x_m)$. The "$p$" in the superscript will indicate that, it is the p-th proposed estimator of σ of a particular kind. The "1,1" in the subscript of U and g will indicate that we are forming a U-statistic by comparing each observation with every other observation.

In general, "k, l" in the subscript will indicate that we may compare the average of each subgroup of size k with the average of each subgroup of size l, where k, l are fixed, $k + l = m$. For example, with $m = 4$, we may have either $k = 1, l = 3,$ equivalently, $k = 3, l = 1,$ or $k = l = 2,$ which would lead to two choices of unbiased estimators of θ constructed our way. These will correspond to $p = 1, 2,$ respectively.

Now, since $(n-1)S^2 / \sigma^2 \sim \chi^2_{n-1},$ it is easy to see that

$$E_{\mu, \sigma}[a_n^{-1} S] = \sigma, \text{ where } a_n = \sqrt{\frac{2}{n-1}} \Gamma\left(\frac{n}{2}\right)\left\{\Gamma\left(\frac{n-1}{2}\right)\right\}^{-1}, \; n \geq 2. \tag{2.3}$$

Thus, using our scheme of notation, we will write

$$T^{(1)} \equiv T^{(2,1)}_{n,1,1} = a_n^{-1} S, \tag{2.4}$$

for the 1st unbiased estimator of $\sigma,$ where the U-statistic corresponding to σ^2 employed in the basic construction of S^2 happens to be $U^{(2,1)}_{n,1,1}$ associated with the kernel $g^{(2,1)}_{1,1}(x_1, x_2)$. However, we note that $T^{(2,1)}_{n,1,1}$ is not a U-statistic in itself, but it is the *minimum variance unbiased estimator* (MVUE) of $\sigma.$

Nair [19] had argued that an unbiased estimator constructed from Gini's mean difference, namely, $U^{(2,2)}_{n,1,1}$ corresponding to the kernel $g^{(2,2)}_{1,1}$ $(x_1, x_2) = \frac{\sqrt{\pi}}{2}|x_1 - x_2|,$ may be used for estimating the population standard deviation σ although, it is slightly less reliable than the estimator $T^{(2,1)}_{n,1,1}$ from (2.4) in the case of a normal distribution. He proposed estimating σ with

$$T^{(2)} \equiv T^{(2,2)}_{n,1,1} = U^{(2,2)}_{n,1,1} = \binom{n}{2}^{-1} \sum_{1 \leq i_1 < i_2 \leq n} g^{(2,2)}_{1,1}(X_{i_1}, X_{i_2}). \tag{2.5}$$

2.1. Motivation behind our construction

In view of the constructions seen in (2.2) and (2.5), let us first motivate a kernel of degree $m = 3,$ for unbiasedly estimating σ. In order to come up with an unbiased estimator of $\sigma,$ why do we not begin by comparing $\frac{1}{2}(X_i + X_j)$ with X_k for each triplet $1 \le i < j < k \le n$? We begin with a basic kernel $|\frac{1}{2}(X_i + X_j) - X_k|,$ symmetrize it and then make it unbiased for σ. This leads to the following kernel:

$$g_{n,2,1}^{(3,1)}(x_1, x_2, x_3)$$

$$= \frac{\sqrt{\pi}}{3\sqrt{3}}\{|\frac{1}{2}(x_1 + x_2) - x_3| + |\frac{1}{2}(x_1 + x_3) - x_2| + |\frac{1}{2}(x_3 + x_2) - x_1|\}. \quad (2.6)$$

This kernel leads to the following U-statistic of degree 3:

$$T^{(3)} \equiv T_{n,2,1}^{(3,1)} = U_{n,2,1}^{(3,1)} = \binom{n}{3}^{-1} \sum_{1 \le i_1 < i_2 < i_3 \le n} g_{n,2,1}^{(3,1)}(X_{i_1}, X_{i_2}, X_{i_3}), \quad (2.7)$$

which estimates σ unbiasedly for $n \ge 3$.

We may additionally propose the following two U-statistics of degree 4. Let us define

$$T^{(4)} \equiv T_{n,2,2}^{(4,1)} = U_{n,2,2}^{(4,1)} = \binom{n}{4}^{-1} \sum_{1 \le i_1 < i_2 < i_3 < i_4 \le n} g_{n,2,2}^{(4,1)}(X_{i_1}, X_{i_2}, X_{i_3}, X_{i_4}),$$

$$(2.8)$$

associated with the kernel

$$g_{n,2,2}^{(4,1)}(x_1, x_2, x_3, x_4)$$

$$= \frac{\sqrt{\pi}}{3\sqrt{2}}\{|\frac{1}{2}(x_1 + x_2) - \frac{1}{2}(x_3 + x_4)| + |\frac{1}{2}(x_1 + x_3) - \frac{1}{2}(x_2 + x_4)|$$

$$+ \left| \frac{1}{2}(x_1 + x_4) - \frac{1}{2}(x_2 + x_3) \right| \Big\}, \tag{2.9}$$

and

$$T^{(5)} \equiv T^{(4,2)}_{n,3,1} = U^{(4,2)}_{n,3,1} = \binom{n}{4}^{-1} \sum_{1 \le i_1 < i_2 < i_3 < i_4 \le n} g^{(4,2)}_{n,3,1}(X_{i_1}, X_{i_2}, X_{i_3}, X_{i_4}), \tag{2.10}$$

associated with the kernel

$$g^{(4,2)}_{n,3,1}(x_1, x_2, x_3, x_4)$$

$$= \frac{\sqrt{3\pi}}{8\sqrt{2}} \Big\{ \left| \frac{1}{3}(x_1 + x_2 + x_3) - x_4 \right| + \left| \frac{1}{3}(x_1 + x_2 + x_4) - x_3 \right|$$

$$+ \left| \frac{1}{3}(x_1 + x_3 + x_4) - x_2 \right| + \left| \frac{1}{3}(x_2 + x_3 + x_4) - x_1 \right| \Big\}, \tag{2.11}$$

for $n \ge 4$.

The two U-statistics $T^{(4,1)}_{n,2,2}$ from (2.8) and $T^{(4,2)}_{n,3,1}$ from (2.10) both have degree 4 and both estimate σ unbiasedly for $n \ge 4$. For brevity, we leave out the details for constructing unbiased estimators of σ, more generally, with U-statistics of degree higher than four.

Before we get to the topic of variance calculations for $T^{(2,1)}_{n,1,1}$, $T^{(2,2)}_{n,1,1}$, $T^{(3,1)}_{n,2,1}$, $T^{(4,1)}_{n,2,2}$, and $T^{(4,2)}_{n,3,1}$ from (2.4), (2.5), (2.7), (2.8), and (2.10), respectively, we first estimated their means and standard deviations based on 5000 replications via computer simulations under each configuration of the parent populations. We fixed n = 15, 25, 30, 45, 60, and 70 and considered random samples from the parent populations $N(5,\ 100)$, $N(5,\ 400)$, and $N(5,\ 900)$. Tables 1 and 2, respectively, summarize our findings in the case of small sample sizes, namely, when n = 15, 25, 30, 45 and for medium sample sizes, namely, when n = 60, 70.

In the case of a specific parent population and sample size n, we obtained the values of all five T' S which gave rise to the corresponding observed t-values from the same set of data during the r-th replication, $r = 1, \ldots, 5000$. From such 5000 observed values, we obtained the sample mean values, $\bar{t}$ and their estimated standard errors, $s(\bar{t})$. In Tables 1 and 2, we show $\bar{t}$ and $s(\bar{t})$ values under each estimator.

From Tables 1 and 2, it appears empirically that $\mathrm{Var}[T^{(4)}] < \mathrm{Var}[T^{(3)}] < \mathrm{Var}[T^{(2)}] < \mathrm{Var}[T^{(5)}]$ for sample sizes 15, 25, 30, 45, and $\mathrm{Var}[T^{(4)}] < \mathrm{Var}[T^{(2)}] < \mathrm{Var}[T^{(3)}] < \mathrm{Var}[T^{(5)}]$ for sample sizes 60, 70 with few exceptions. Thus, empirically $T^{(4)}$ appears to be the best estimator among $T^{(2)}$ through $T^{(5)}$. In fact, the asymptotic variance of $T^{(4)}$ is also the smallest among the asymptotic variances of $T^{(2)}$ through $T^{(5)}$.

In summary, from these tables, it is abundantly clear to us that the three new unbiased estimators $T^{(3)}, T^{(4)}$, and $T^{(5)}$ for σ; (i) go practically head-to-head with existing estimators $T^{(1)}$ and $T^{(2)}$, (ii) $T^{(4)}$ beats $T^{(2)}$, and (iii) $T^{(3)}$ very nearly beats $T^{(2)}$, whether n is small or moderately large.

Table 1. Simulated estimates $(\bar{t})$ of standard deviation and their corresponding standard error $s(\bar{t})$

	n	$\bar{t}^{(1)}$ / $s(\bar{t}^{(1)})$	$\bar{t}^{(2)}$ / $s(\bar{t}^{(2)})$	$\bar{t}^{(3)}$ / $s(\bar{t}^{(3)})$	$\bar{t}^{(4)}$ / $s(\bar{t}^{(4)})$	$\bar{t}^{(5)}$ / $s(\bar{t}^{(5)})$
$\boldsymbol{N}$**(5, 100)**	15	10.00007	9.99426	9.99462	9.99964	9.99312
		1.92789	1.94194	1.93883	1.92751	1.94833
	25	9.96733	9.96803	9.96752	9.96675	9.96800
		1.45405	1.47153	1.47107	1.45638	1.47970
	30	10.00541	9.99965	9.99955	10.00317	9.99815
		1.32520	1.33483	1.33312	1.32488	1.33890
	45	9.99183	9.99473	9.99487	9.99295	9.99537
		1.08470	1.09513	1.09493	1.08657	1.10074
$\boldsymbol{N}$**(5, 400)**	15	20.01364	20.00570	20.00462	20.01027	20.00137
		3.77603	3.80948	3.80431	3.77453	3.82597
	25	20.02791	20.02587	20.02529	20.02726	20.02375
		2.90472	2.93433	2.93066	2.90614	2.94621
	30	20.04563	20.04681	20.04494	20.04653	20.04365
		2.61868	2.64368	2.64282	2.62143	2.65648
	45	20.04855	20.04327	20.04279	20.04632	20.04254
		2.16611	2.19184	2.18960	2.17070	2.20495
$\boldsymbol{N}$**(5, 900)**	15	29.97534	29.98079	29.98705	29.97867	29.98913
		5.70799	5.77802	5.77015	5.71273	5.80529
	25	30.02527	30.02623	30.02265	30.02438	30.02220
		4.43593	4.47374	4.47062	4.43833	4.49376
	30	29.96168	29.95199	29.95250	29.95972	29.94997
		4.00480	4.04125	4.04104	4.00851	4.06237
	45	30.02106	30.02924	30.02785	30.02461	30.02856
		3.22111	3.26143	3.26199	3.23030	3.28064

Table 2. Simulated estimates $(\bar{t})$ of standard deviation and their corresponding standard error $s(\bar{t})$

	n	$\bar{t}^{(1)}$ / $s(\bar{t}^{(1)})$	$\bar{t}^{(2)}$ / $s(\bar{t}^{(2)})$	$\bar{t}^{(3)}$ / $s(\bar{t}^{(3)})$	$\bar{t}^{(4)}$ / $s(\bar{t}^{(4)})$	$\bar{t}^{(5)}$ / $s(\bar{t}^{(5)})$
***N*(5, 100)**	60	9.99572	9.99712	9.99775	9.99620	9.99844
		0.93330	0.94144	0.94150	0.93405	0.94623
	70	9.99685	9.99688	9.99721	9.99684	9.99747
		0.84230	0.84940	0.84981	0.84292	0.85414
***N*(5, 400)**	60	20.00012	19.99386	19.99382	19.99774	19.99232
		1.81046	1.82902	1.82886	1.81339	1.83817
	70	19.99676	19.99319	19.99331	19.99494	19.99295
		1.71972	1.73420	1.73461	1.72176	1.74287
***N*(5, 900)**	60	30.05758	30.05713	30.05611	30.05751	30.05457
		2.75006	2.79284	2.79353	2.76022	2.81196
	70	30.02919	30.03386	30.03391	30.03081	30.03533
		2.51252	2.54109	2.54258	2.51822	2.55735

3. Exact Variances of $T^{(1)}$, $T^{(2)}$, $T^{(3)}$ and also Large-Sample Approximations of the Variances of $T^{(1)}$ Through $T^{(5)}$

One can easily verify that the exact variance of the unbiased form of S, namely, $T^{(1)}$ from (2.4), is given by

$$\operatorname{Var}[T^{(1)}] = (a_n^{-2} - 1)\sigma^2 \text{ with } a_n = \sqrt{2}\Gamma\left(\frac{n}{2}\right)\left\{\Gamma\left(\frac{n-1}{2}\right)\sqrt{n-1}\right\}^{-1}. \tag{3.1}$$

Nair [19] had shown that the variance of $T^{(2)}$ defined in (2.5), is given by

$$\operatorname{Var}[T^{(2)}] = \frac{1}{n(n-1)}\left\{\frac{\pi}{3}(n+1) + 2(n-2)\sqrt{3} - 2(2n-3)\right\}\sigma^2. \tag{3.2}$$

The exact variance of $T^{(3)}$, defined in (2.7), is given by

$$\operatorname{Var}[T^{(3)}] = \binom{n}{3}^{-1}\left\{3\binom{n-3}{2}\zeta_1^{(3)} + 3(n-3)\zeta_2^{(3)} + \zeta_3^{(3)}\sigma^2\right\}, \tag{3.3}$$

where we denote

$$\zeta_1^{(3)} = \frac{1}{27}[2\sqrt{35} + 2\sin^{-1}\left(\frac{1}{6}\right) + \sqrt{5} + 2\sin^{-1}\left(\frac{2}{3}\right)$$

$$+ 8\sqrt{2} + 4\sin^{-1}\left(\frac{1}{3}\right)] - 1; \tag{3.4}$$

$$\zeta_2^{(3)} = \frac{1}{27}[2\sqrt{2} + \sin^{-1}\left(\frac{1}{3}\right) + \sqrt{11} + 5\sin^{-1}\left(\frac{5}{6}\right) + 2\sqrt{35}$$

$$+ 2\sin^{-1}\left(\frac{1}{6}\right) + 2\sqrt{5} + 4\sin^{-1}\left(\frac{2}{3}\right)] - 1; \tag{3.5}$$

and

$$\zeta_3^{(3)} = \left[\frac{\pi}{6} + \frac{1}{\sqrt{3}} + \frac{1}{3}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) - 1\right]. \tag{3.6}$$

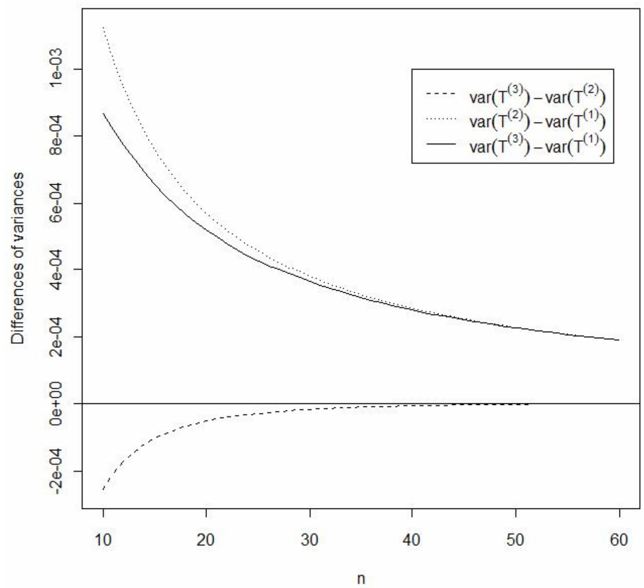
A proof of (3.3) is given in the subsection 4.

3.1. Comparing the exact variances

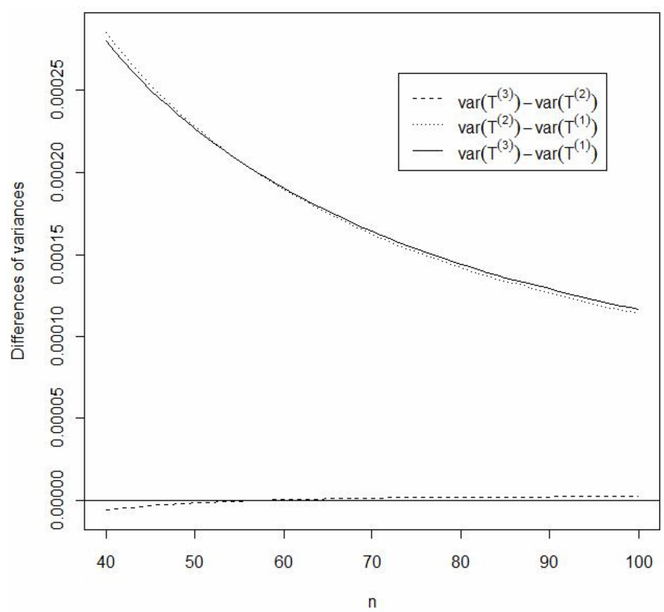
In Figures 1(a) and 1(b), we have presented the curves corresponding to $\mathrm{Var}[T^{(3)}] - \mathrm{Var}[T^{(2)}]$ vs. n (solid), $\mathrm{Var}[T^{(2)}] - \mathrm{Var}[T^{(1)}]$ vs. n (dashed), and $\mathrm{Var}[T^{(3)}] - \mathrm{Var}[T^{(1)}]$ vs. n (dotted) when $n = 10(1)60$ and $n = 40(1)100$, respectively. In Figures 2(a) and 2(b), we have additionally presented the curves corresponding to the efficiency of each estimator relative to another vs. n when $n = 10(1)60$ and $n = 40(1)100$, respectively. The efficiency of an estimator $T^{(i)}$ relative to another estimator $T^{(j)}$ is traditionally quantified by

$$\text{Efficiency, } \mathrm{Eff}_{ij} = \frac{\mathrm{Var}[T^{(j)}]}{\mathrm{Var}[T^{(i)}]}. \tag{3.7}$$

Figures 2(a) and 2(b) plot the curves Eff_{23} (solid), Eff_{12} (dashed), and Eff_{13} (dotted) vs. n when $n = 10(1)60$ and $n = 40(1)100$, respectively. We have deliberately kept a small overlap of n-values when we move from Figure (a) to Figure (b) in order to be able to pick up any visible differences when $n = 40(1)60$.

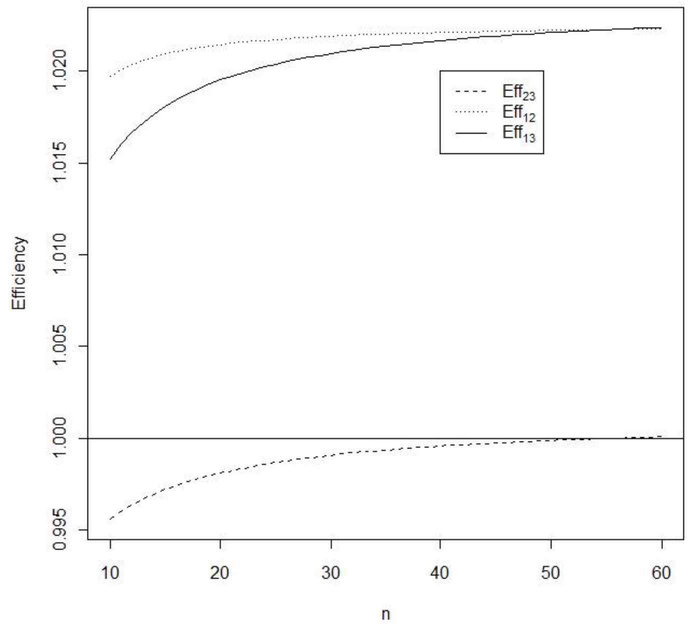


(a)

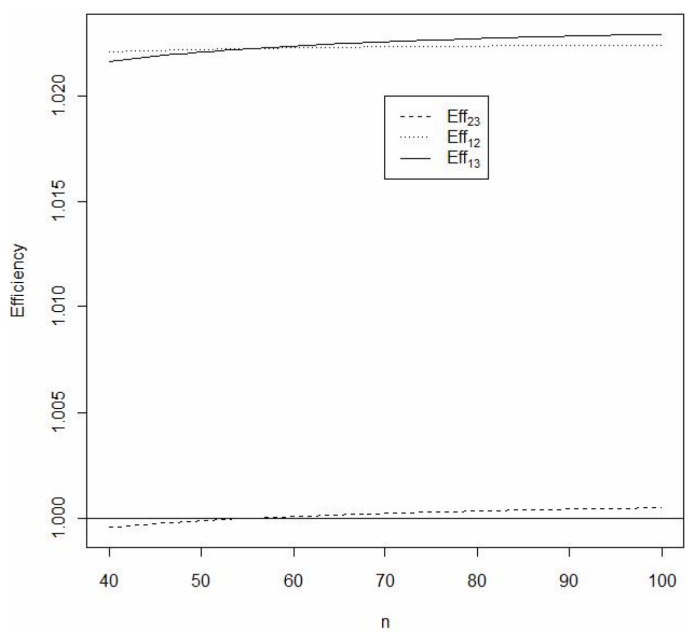


(b)

Figure 1. Differences of variances from (3.1)-(3.3): $\mathrm{Var}[T^{(3)}] - \mathrm{Var}[T^{(2)}]$ (solid), $\mathrm{Var}[T^{(2)}] - \mathrm{Var}[T^{(1)}]$ (dashed), and $\mathrm{Var}[T^{(3)}] - \mathrm{Var}[T^{(1)}]$ (dotted) vs. n; (a) n = 10(1)60 and (b) n = 40(1)100.



(a)



(b)

Figure 2. Efficiencies from (3.7): Eff_{23} (solid), Eff_{12} (dashed), and Eff_{13} (dotted) vs. n; (a) n = 10(1)60 and (b) n = 40(1)100.

From the Figures 1(a), 1(b), 2(a), and 2(b), we note that for a reasonably small sample size (that is, when $n \leq 55$), the estimator $T^{(3)}$ is more efficient than $T^{(2)}$. For a sample of size exceeding 55, the estimator $T^{(2)}$ is more efficient than $T^{(3)}$. One may note that when $n = 3$, $T^{(2)}$ and $T^{(3)}$ have equal variances.

Clearly, the unbiased version of the sample standard deviation, namely, $T^{(1)}$, is more efficient than either $T^{(2)}$ or $T^{(3)}$ whatever is the sample size. Thus, if we use $T^{(3)}$ in place of $T^{(2)}$ (when $n > 55$) or instead of $T^{(1)}$ (whatever be n) for estimating σ, then one will obviously encounter some loss of information. From Figures 2(a) and 2(b), we surmise that the loss of efficiency, while using $T^{(3)}$ in place of $T^{(2)}$ when $n > 55$ will not exceed 1.5×10^{-3} whatever may be n, whereas the loss of efficiency for using $T^{(3)}$ instead of $T^{(1)}$ will not exceed 0.025 whatever is n. Thus, one may not lose much information, if one uses $T^{(3)}$ instead of $T^{(2)}$ or $T^{(1)}$.

3.2. Large-sample comparisons of the variances of estimators

In the Section 4, we have provided large-sample approximations for the variances of the estimators $T^{(1)}$ through $T^{(5)}$. In what follows, we summarize these large-sample approximations of $\operatorname{Var}[T^{(1)}]$ through $\operatorname{Var}[T^{(5)}]$.

$$\operatorname{Var}[T^{(1)}] \approx 0.5\frac{\sigma^2}{n}; \operatorname{Var}[T^{(2)}] \approx 0.5112992\frac{\sigma^2}{n}; \operatorname{Var}[T^{(3)}] \approx 0.5118784\frac{\sigma^2}{n};$$

$$\operatorname{Var}[T^{(4)}] \approx 0.5026544\frac{\sigma^2}{n}; \text{ and } \operatorname{Var}[T^{(5)}] \approx 0.5172479\frac{\sigma^2}{n}. \quad (3.8)$$

As expected, $T^{(1)}$ is most efficient among our five competitive unbiased estimators of σ. But, for large sample sizes, $T^{(2)}$ through $T^{(4)}$

are not terribly worse than $T^{(1)}$, whereas (i) $T^{(4)}$ clearly stands out especially since it can beat $T^{(2)}$, and (ii) $T^{(3)}$ performs nearly head-to-head with $T^{(2)}$.

4. Proofs and Derivations

In this section, we provide some of the major derivations. In Subsections 4.1-4.4, we have repeatedly utilized Lemmas 4.1-4.6, which have been kept out of the way by combining all of them as well as their proofs in Subsection 4.5.

4.1. Large-sample variance of $T^{(1)}$ from (3.8)

The exact expression of the variance of $T^{(1)}$ in (3.1) is easy to verify. From Abramowitz and Stegun ([1], 6.1.47), first we know that

$$z^{b-a}\,\frac{\Gamma(z+a)}{\Gamma(z+b)} \sim 1 + \frac{1}{2z}(a-b)(a+b-1) + 0(z^{-2}). \tag{4.1}$$

Now, we rewrite $\text{Var}[T^{(1)}]$ from (3.1) as

$$\begin{aligned}(a_n^{-2}-1)\sigma^2 &= \left\{\frac{n-1}{2}\left(\Gamma\left(\frac{n-1}{2}\right)\right)^2\left(\Gamma\left(\frac{n}{2}\right)\right)^2 - 1\right\}\sigma^2 \\ &= \frac{1}{2n}\sigma^2 + 0(n^{-2}),\end{aligned}$$

in view of (4.1).

4.2. Exact variance of $T^{(3)}$: Proof of (3.3)

Let us first consider a general U-statistic U defined by (2.1). Then, using Hoeffding's [10] projection method, the variance of U will be given by

$$\text{Var}[U] = \binom{n}{m}^{-1}\sum_{c=1}^{m}\binom{m}{c}\binom{n-m}{m-c}\zeta_c,$$

where

$$\zeta_c = E\left[g^{(m)2}(X_{i_1}, \dots, X_{i_m})|X_{i_1} = x_{i_1}, \dots, X_{i_c} = x_{i_c}\right] - \theta^2, \qquad (4.2)$$

for $c = 1, 2, \dots, m$.

Next, in view of (2.7), we have

$$T^{(3)} = \binom{n}{3}^{-1} \sum_{1 \le i < j < k \le n} \sqrt{\frac{\pi}{27}} \left\{ \left| \frac{X_i + X_j}{2} - X_k \right| + \left| \frac{X_k + X_i}{2} - X_j \right| \right.$$

$$\left. + \left| \frac{X_j + X_k}{2} - X_i \right| \right\}. \qquad (4.3)$$

So, we let $W_1 = \frac{X_1 + X_2}{2} - X_3$, $W_2 = \frac{X_1 + X_3}{2} - X_2$, and $W_3 = \frac{X_2 + X_3}{2} - X_1$ with $W_i \sim N\left(0, \frac{3}{2}\sigma^2\right)$, for $i = 1, 2, 3$. But, we know the following conditional distributions:

$$W_1|X_1 \sim N\left(\frac{X_1 - \mu}{2}, \frac{5}{4}\sigma^2\right) \text{ and } W_1|X_3 \sim N\left(\frac{X_3 - \mu}{2}, \frac{5}{4}\sigma^2\right). \qquad (4.4)$$

Thus, utilizing (4.4), we can claim

$$I_{11} = I_{12} = E[|W_1||X_1] = \sigma\left\{\sqrt{\frac{5}{2\pi}} \exp\left(-\frac{1}{10}Z^2\right) + Z\Phi\left(\frac{1}{\sqrt{5}}Z\right) - \frac{1}{2}Z\right\},$$

$$I_{13} = E[|W_3||X_1] = \sigma\left\{\frac{1}{\sqrt{\pi}} \exp(-Z^2) + 2Z\Phi\left(\sqrt{2}Z\right) - Z\right\}, \qquad (4.5)$$

where $Z = (X_1 - \mu)/\sigma$.

Also, we know the following conditional distributions:

$$W_1|X_1, X_2 \sim N\left(\frac{X_1 + X_2}{2} - \mu, \sigma^2\right) \text{ and } W_1|X_1, X_3 \sim N\left(\frac{X_1 - 2X_3 + \mu}{2}, \frac{1}{4}\sigma^2\right).$$

$$(4.6)$$

Utilizing (4.6), we obtain

$$
\begin{aligned}
I_{21} &= E\left[|W_1|\,|X_1,\, X_2\right] \\
&= \sigma\left\{\sqrt{\frac{2}{\pi}}\exp\left(-\frac{Z_1^2}{4}\right) + \sqrt{2}Z_1\Phi\left(\frac{Z_1}{\sqrt{2}}\right) - \frac{Z_1}{\sqrt{2}}\right\},\ Z_1 = \frac{X_1 + X_2 - 2\mu}{\sqrt{2}\sigma}, \\
I_{22} &= E\left[|W_2|\,|X_1,\, X_2\right] \\
&= \sigma\left\{\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{5Z_2^2}{2}\right) + \sqrt{5}Z_2\Phi\left(\sqrt{5}Z_2\right) - \frac{\sqrt{5}Z_2}{2}\right\},\ Z_2 = \frac{X_1 - 2X_2 + \mu}{\sqrt{5}\sigma}, \\
I_{23} &= E\left[|W_3|\,|X_1,\, X_2\right] \\
&= \sigma\left\{\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{5Z_3^2}{2}\right) + \sqrt{5}Z_3\Phi\left(\sqrt{5}Z_3\right) - \frac{\sqrt{5}Z_3}{2}\right\},\ Z_3 = \frac{X_2 - 2X_1 + \mu}{\sqrt{5}\sigma}.
\end{aligned}
\tag{4.7}
$$

We also observe that $E[I_{11}^2] = E[I_{12}^2] = E[I_{11}I_{12}]$ and we can evaluate this term as follows:

$$
\begin{aligned}
&\sigma^2\left[\frac{5}{2\pi}E\left\{\exp\left(-\frac{Z^2}{5}\right)\right\} + \frac{1}{4}E(Z^2) + E\left\{Z^2\Phi^2\left(\frac{Z}{\sqrt{5}}\right)\right\} - E\left\{Z^2\Phi\left(\frac{Z}{\sqrt{5}}\right)\right\}\right. \\
&\qquad \left. - \sqrt{\frac{5}{2\pi}}E\left\{Z\exp\left(-\frac{Z^2}{10}\right)\right\} + 2\sqrt{\frac{5}{2\pi}}E\left\{Z\exp\left(-\frac{Z^2}{10}\right)\Phi\left(\frac{Z}{\sqrt{5}}\right)\right\}\right] \\
&= \sigma^2\left(\frac{5}{2\pi}\sqrt{\frac{5}{7}} + \frac{1}{4} + \left[\frac{1}{4} + \frac{1}{2\pi}\sin^{-1}\left(\frac{1}{6}\right) + \frac{\sqrt{5}}{6\pi\sqrt{7}}\right] - \frac{1}{2} + \frac{5\sqrt{5}}{6\pi\sqrt{7}}\right) \\
&= \sigma^2\left(\frac{\sqrt{35}}{2\pi} + \frac{1}{2\pi}\sin^{-1}\left(\frac{1}{6}\right)\right).
\end{aligned}
\tag{4.8}
$$

The terms such as

$$
E\left(Z^2\Phi^2\left(Z/\sqrt{5}\right)\right),\ E\left(Z^2\Phi\left(Z/\sqrt{5}\right)\right),\ \text{and}\ E\left\{Z\exp(-Z^2/10)\Phi\left(Z/\sqrt{5}\right)\right\}
$$

involved in (4.8) were evaluated by appealing to Lemmas 4.6, 4.2, and 4.5.

Next, we turn to evaluate $E[I_{13}^2]$ and $E[I_{11}I_{13}](= E[I_{12}I_{13}])$. We have

$$E[I_{13}^2] = \sigma^2\Big[\frac{1}{\pi}E\big(\exp(-2Z^2)\big) + 4E\big(Z^2\Phi^2(\sqrt{2}Z)\big) - 4E\big(Z^2\Phi(\sqrt{2}Z)\big)$$

$$+ E(Z^2) - \frac{2}{\sqrt{\pi}}E\big(Z\exp(-Z^2)\big) + \frac{4}{\sqrt{\pi}}E\big(Z\exp(-Z^2)\Phi(\sqrt{2}Z)\big)\Big]$$

$$= \sigma^2\left[\frac{1}{\pi\sqrt{5}} + 4\left(\frac{1}{4} + \frac{1}{2\pi}\sin^{-1}\left(\frac{2}{3}\right) + \frac{2}{3\pi\sqrt{5}}\right) - 1 + \frac{4}{\sqrt{\pi}}\cdot\frac{1}{3\sqrt{5\pi}}\right]$$

$$= \sigma^2\left(\frac{\sqrt{5}}{\pi} + \frac{2}{\pi}\sin^{-1}\left(\frac{2}{3}\right)\right), \tag{4.9}$$

and

$$E[I_{11}I_{13}] = E[I_{12}I_{13}]$$

$$= \sigma^2\left[\frac{1}{\pi}\sqrt{\frac{5}{2}}E\left(\exp\left(-\frac{11Z^2}{10}\right)\right) + 2E\left(Z^2\Phi\left(\frac{Z}{\sqrt{5}}\right)\Phi(\sqrt{2}Z)\right)\right.$$

$$-\sqrt{\frac{5}{2\pi}}E\left(Z\exp\left(-\frac{Z^2}{10}\right)\right) - \frac{1}{2\sqrt{\pi}}E(Z\exp(-Z^2)) + \frac{1}{2}E(Z^2)$$

$$+ 2\sqrt{\frac{5}{2\pi}}E\left(Z\exp\left(-\frac{Z^2}{10}\right)\Phi(\sqrt{2}Z)\right) - E\left(Z^2\Phi\left(\frac{Z}{\sqrt{5}}\right)\right)$$

$$\left.+ \frac{1}{\sqrt{\pi}}E\left(Z\Phi\left(\frac{Z}{\sqrt{5}}\right)\exp(-Z^2)\right) - E(Z^2\Phi(\sqrt{2}Z))\right]$$

$$= \sigma^2\left[\frac{1}{\pi}\sqrt{\frac{5}{2}}\cdot\frac{\sqrt{5}}{4} + 2\left(\frac{1}{4} + \frac{1}{2\pi}\sin^{-1}\left(\frac{1}{3}\right) + \frac{5\sqrt{2}}{48\pi} + \frac{\sqrt{2}}{24\pi}\right)\right.$$

$$\left.+ \frac{1}{2} + 2\sqrt{\frac{5}{2\pi}}\cdot\frac{5\sqrt{5}}{24\sqrt{\pi}} - \frac{1}{2} + \frac{1}{\sqrt{\pi}}\,\frac{1}{12\sqrt{2\pi}} - \frac{1}{2}\right]$$

$$= \sigma^2\left[\frac{2\sqrt{2}}{\pi} + \frac{1}{\pi}\sin^{-1}\left(\frac{1}{3}\right)\right]. \tag{4.10}$$

Again, the terms such as

$$E\left(Z^2\Phi^2\left(\sqrt{2}Z\right)\right),\ E\left(Z^2\Phi\left(\sqrt{2}Z\right)\right),\ \text{and}\ E\left(Z\exp(-Z^2)\Phi\left(\sqrt{2}Z\right)\right)$$

involved in (4.9) as well as similar terms involved in (4.10) were evaluated by appealing to Lemmas 4.6, 4.2, and 4.5 as needed.

Also, we can evaluate $E[I_{21}^2]$ and $E[I_{22}^2] = E[I_{23}^2]$ as follows:

$$\begin{aligned}
E[I_{21}^2] = \sigma^2\Bigg[& \frac{2}{\pi}E\left\{\exp\left(-\frac{Z_1^2}{2}\right)\right\} + \frac{1}{2}E(Z_1^2) + 2E\left\{Z_1^2\Phi^2\left(\frac{Z_1}{\sqrt{2}}\right)\right\} \\
& + \frac{4}{\sqrt{\pi}}E\left\{Z_1\exp\left(-\frac{Z_1^2}{4}\right)\Phi\left(\frac{Z_1}{\sqrt{2}}\right)\right\} - 2E\left\{Z_1^2\Phi\left(\frac{Z_1}{\sqrt{2}}\right)\right\} \\
& - \frac{2}{\sqrt{\pi}}E\left\{Z_1\exp\left(-\frac{Z_1^2}{4}\right)\right\}\Bigg] \\
= \sigma^2\Bigg[& \frac{\sqrt{2}}{\pi} + \frac{1}{2} + 2\left\{\frac{1}{4} + \frac{1}{2\pi}\sin^{-1}\left(\frac{1}{3}\right) + \frac{1}{3\pi\sqrt{2}}\right\} + \frac{4}{\sqrt{\pi}}\cdot\frac{1}{3\sqrt{2\pi}} - 1\Bigg] \\
= \sigma^2\Bigg[& \frac{2\sqrt{2}}{\pi} + \frac{1}{\pi}\sin^{-1}\left(\frac{1}{3}\right)\Bigg],
\end{aligned} \tag{4.11}$$

and

$$\begin{aligned}
E[I_{22}^2] &= E[I_{23}^2] \\
&= \sigma^2\Bigg[\frac{1}{2\pi}E\{\exp(-5Z_2^2)\} + \frac{5}{4}E[Z_2^2] + 5E\{Z_2^2\Phi^2(\sqrt{5}Z_2)\} \\
&\quad + \sqrt{\frac{10}{\pi}}E\{Z_2\exp(-\frac{5}{2}Z_2^2)\Phi(\sqrt{5}Z_2)\} - 5E\{Z_2^2\Phi(\sqrt{5}Z_2)\} \\
&\quad - \sqrt{\frac{5}{2\pi}}E\{Z_2\exp(-\frac{5}{2}Z_2^2)\}\Bigg] \\
&= \sigma^2\Bigg[\frac{1}{2\pi}\cdot\frac{1}{\sqrt{11}} + \frac{5}{4} + 5\left\{\frac{1}{4} + \frac{1}{2\pi}\sin^{-1}(\frac{5}{6}) + \frac{5}{6\pi\sqrt{11}}\right\}
\end{aligned}$$

$$+\sqrt{\frac{10}{\pi}}\frac{\sqrt{5}}{6\sqrt{22\pi}}-\frac{5}{2}\Bigg]$$

$$=\sigma^2\left[\frac{\sqrt{11}}{2\pi}+\frac{5}{2\pi}\sin^{-1}\left(\frac{5}{6}\right)\right]. \tag{4.12}$$

The terms such as

$$E\left(Z_1^2\Phi^2\left(Z_1/\sqrt{2}\right)\right),\ E\left(Z_1\exp(-Z_1^2/4)\Phi\left(Z_1/\sqrt{2}\right)\right),\text{ and } E\left(Z_1^2\Phi\left(Z_1/\sqrt{2}\right)\right)$$

involved in (4.11) as well as similar terms involved in (4.12) were evaluated by appealing to Lemmas 4.6, 4.2, and 4.5 as needed.

Next, we turn to evaluate $E[I_{21}I_{22}]$, equivalently $E[I_{21}I_{23}]$, and $E[I_{22}I_{23}]$. We write

$$E[I_{21}I_{22}]=E[I_{21}I_{23}]$$

$$=\sigma^2\Bigg[\frac{1}{\pi}E\left\{\exp\left(-\frac{Z_1^2}{4}-\frac{5Z_2^2}{2}\right)\right\}+\frac{1}{\sqrt{\pi}}E\left\{Z_1\Phi\left(\frac{Z_1}{\sqrt{2}}\right)\exp\left(-\frac{5}{2}Z_2^2\right)\right\}$$

$$-\frac{1}{2\sqrt{\pi}}E\{Z_1\exp(-\frac{5}{2}Z_2^2)\}+\sqrt{\frac{10}{\pi}}E\left\{Z_2\exp\left(-\frac{Z_1^2}{4}\right)\Phi(\sqrt{5}Z_2)\right\}$$

$$-\sqrt{\frac{5}{2}}E\{Z_1Z_2\Phi(\sqrt{5}Z_2)\}+\sqrt{10}E\left\{Z_1Z_2\Phi\left(\frac{Z_1}{\sqrt{2}}\right)\Phi(\sqrt{5}Z_2)\right\}$$

$$-\sqrt{\frac{5}{2}}E\left\{Z_1Z_2\Phi\left(\frac{Z_1}{\sqrt{2}}\right)\right\}-\sqrt{\frac{5}{2\pi}}E\left\{Z_2\exp\left(-\frac{Z_1^2}{4}\right)\right\}+\frac{\sqrt{5}}{2\sqrt{2}}E(Z_1Z_2)\Bigg]$$

$$=\sigma^2\left[\frac{2}{\sqrt{35}\pi}+\frac{11}{12\pi\sqrt{35}}+\frac{29}{3\pi\sqrt{35}}+\frac{33}{8\pi\sqrt{35}}+\frac{1}{2\pi}\sin^{-1}\left(\frac{1}{6}\right)+\frac{19}{24\pi\sqrt{35}}\right]$$

$$=\sigma^2\left[\frac{\sqrt{35}}{2\pi}+\frac{1}{2\pi}\sin^{-1}\left(\frac{1}{6}\right)\right], \tag{4.13}$$

and

$$E[I_{22}I_{23}]$$

$$= \sigma^2\left[\frac{1}{2\pi} E\left\{\exp\left(-\frac{5Z_2^2}{2} - \frac{5Z_3^2}{2}\right)\right\} + \sqrt{\frac{5}{2\pi}} E\{Z_2 \Phi(\sqrt{5} Z_2) \exp(-\frac{5}{2} Z_3^2)\}\right.$$

$$- \frac{\sqrt{5}}{2\sqrt{2\pi}} E\{Z_2 \exp(-\frac{5}{2} Z_3^2)\} + \sqrt{\frac{5}{2\pi}} E\left\{Z_3 \exp\left(-\frac{5Z_2^2}{2}\right)\Phi(\sqrt{5} Z_3)\right\}$$

$$- \frac{5}{2} E\{Z_2 Z_3 \Phi(\sqrt{5} Z_3)\} + 5E\{Z_2 Z_3 \Phi(\sqrt{5} Z_2) \Phi(\sqrt{5} Z_3)\}$$

$$\left.- \frac{5}{2} E\{Z_2 Z_3 \Phi(\sqrt{5} Z_2)\} - \frac{\sqrt{5}}{2\sqrt{2\pi}} E\left\{Z_3 \exp\left(-\frac{5Z_2^2}{2}\right)\right\} + \frac{5}{4} E(Z_2 Z_3)\right]$$

$$= \sigma^2\left[\frac{1}{4\pi\sqrt{5}} + \frac{7}{6\pi\sqrt{5}} + \frac{2}{\pi}\sin^{-1}\left(\frac{2}{3}\right) + \frac{43}{12\pi\sqrt{5}}\right]$$

$$= \sigma^2\left[\frac{\sqrt{5}}{\pi} + \frac{2}{\pi}\sin^{-1}\left(\frac{2}{3}\right)\right]. \tag{4.14}$$

Recall that evaluations of terms such as

$$E(Z_1 \Phi(Z_1 / \sqrt{2}) \exp(-5Z_2^2 / 2)),\ E(Z_2 \exp(-Z_1^2 / 4) \Phi(\sqrt{5} Z_2)),$$

$$E(Z_1 Z_2 \Phi(\sqrt{5} Z_2)),\ E(Z_1 Z_2 \Phi(Z_1 / \sqrt{2}) \Phi(\sqrt{5} Z_2)),$$

$$\text{and } E(Z_1 Z_2 \Phi(Z_1 / \sqrt{2}))$$

involved in (4.13) as well as similar terms involved in (4.14) use Lemmas 4.6, 4.2, and 4.5 as needed.

Now, then, we have

$$\zeta_1^{(3)} = E\left[\frac{\sqrt{\pi}}{3\sqrt{3}} E(I_{11} + I_{12} + I_{13})\right]^2 - \sigma^2,$$

$$\zeta_2^{(3)} = E\left[\frac{\sqrt{\pi}}{3\sqrt{3}} E(I_{21} + I_{22} + I_{23})\right]^2 - \sigma^2, \tag{4.15}$$

and

$$\zeta_3^{(3)} + \sigma^2 = \frac{\pi}{27} E[|W_1| + |W_2| + |W_3|]^2$$

$$= \frac{\pi}{27} [E(W_1^2) + E(W_2^2) + 2E|W_1W_2| + E(W_3^2)$$

$$+ 2E|W_1W_3| + 2E|W_2W_3|]$$

$$= \frac{\pi}{27} [3E(W_1^2) + 9\sigma^2 E(| \frac{2W_1W_2}{3\sigma^2} |)]. \tag{4.16}$$

Let us denote $Y_1 = \frac{\sqrt{2}W_1}{\sigma\sqrt{3}}$ and $Y_2 = \frac{\sqrt{2}W_2}{\sigma\sqrt{3}}$, so that (4.16) leads to

$$\zeta_3^{(3)} = \frac{\pi}{27} \sigma^2 [\frac{9}{2} + 9E(|Y_1Y_2|)] - \sigma^2$$

$$= \frac{\pi}{3} \sigma^2 \left[\frac{1}{2} + \frac{\sqrt{3}}{\pi} + \frac{1}{\pi} \tan^{-1}\left(\frac{1}{\sqrt{3}} \right) \right] - \sigma^2$$

$$= \sigma^2 \left[\frac{\pi}{6} + \frac{1}{\sqrt{3}} + \frac{1}{3} \tan^{-1}\left(\frac{1}{\sqrt{3}} \right) - 1 \right]. \tag{4.17}$$

The term $E(|Y_1Y_2|)$ involved in (4.17) was evaluated by appealing to Lemma 4.4. Thus, (3.3) follows, that is, the exact variance of $T^{(3)}$ is given by

$$\binom{n}{3}^{-1} \left\{ \binom{3}{1}\binom{n-3}{2}\zeta_1^{(3)} + \binom{3}{2}\binom{n-3}{1}\zeta_2^{(3)} + \binom{3}{3}\binom{n-3}{0}\zeta_3^{(3)} \right\}. \tag{4.18}$$

From (4.18), it also immediately follows that the variance of $T^{(3)}$ is $\frac{9}{n}\zeta_1^{(3)} + O(n^{-1})$ for large n.

4.3. Large-sample variance of $T^{(4)}$

We define $W_{41} = \frac{1}{2}(X_1 + X_2) - \frac{1}{2}(X_3 + X_4)$, so that $W_{41}|X_1 \sim N(\frac{1}{2}(X_1 - \mu), \frac{3}{4}\sigma^2)$. Let $Z = \frac{X_1 - \mu}{\sigma}$, $z = \frac{x_1 - \mu}{\sigma}$. Thus, we can write

$$
\begin{aligned}
I_{41} &= E\left[g_{n,2,2}^{(4,1)}(x_1,\ X_2,\ X_3,\ X_4)\right] \\
&= \frac{\sqrt{\pi}}{3\sqrt{2}}\{E[|\frac{1}{2}(x_1+X_2)-\frac{1}{2}(X_3+X_4)|] \\
&\quad + E[|\frac{1}{2}(x_1+X_3)-\frac{1}{2}(X_2+X_4)|] \\
&\quad + E[|\frac{1}{2}(x_1+X_4)-\frac{1}{2}(X_2+X_3)|]\} \\
&= \sqrt{\frac{\pi}{2}}E[|\frac{1}{2}(x_1+X_2)-\frac{1}{2}(X_3+X_4)|] \\
&= \sqrt{\frac{\pi}{2}}\sigma\left[\sqrt{\frac{3}{2\pi}}\exp(-\frac{1}{6}z^2)+z\Phi\left(\frac{1}{\sqrt{3}}z\right)-\frac{1}{2}z\right]. \qquad (4.19)
\end{aligned}
$$

From (4.19), we obtain

$$
\begin{aligned}
E[I_{41}^2] &= \frac{\pi}{2}\sigma^2E\left[\frac{3}{2\pi}\exp(-\frac{1}{3}Z^2)+Z^2\Phi^2\left(\frac{1}{\sqrt{3}}Z\right)-\sqrt{\frac{3}{2\pi}}Z\exp(-\frac{1}{6}Z^2)\right. \\
&\quad \left. -Z^2\Phi\left(\frac{1}{\sqrt{3}}Z\right)+\sqrt{\frac{6}{\pi}}Z\exp(-\frac{1}{6}Z^2)\Phi\left(\frac{1}{\sqrt{3}}Z\right)+\frac{1}{4}Z^2\right] \\
&= \frac{\pi}{2}\sigma^2\left[\frac{3\sqrt{3}}{2\pi\sqrt{5}}+\frac{1}{2\pi}\sin^{-1}(\frac{1}{4})+\frac{\sqrt{3}}{4\pi\sqrt{5}}+\frac{3\sqrt{3}}{4\pi\sqrt{5}}\right] \\
&= \sigma^2\left[\frac{\sqrt{15}}{4}+\frac{1}{4}\sin^{-1}(\frac{1}{4})\right]. \qquad (4.20)
\end{aligned}
$$

Again, the terms

$$E\left(Z^2\Phi^2\left(Z/\sqrt{3}\right)\right),\ E\left(Z^2\Phi\left(Z/\sqrt{3}\right)\right), \text{ and } E\left(Z\exp(-Z^2/6)\Phi\left(Z/\sqrt{3}\right)\right)$$

involved in (4.20) were evaluated by appealing to Lemmas 4.6, 4.2, and 4.5 as needed.

Thus, from (4.20), we have

$$\zeta_1^{(4)} = \sigma^2\left[\frac{\sqrt{15}}{4} + \frac{1}{4}\sin^{-1}\left(\frac{1}{4}\right) - 1\right]. \tag{4.21}$$

Hence, the asymptotic variance of $T^{(4)}$ is $\dfrac{16}{n}\zeta_1^{(4)} + O(n^{-1})$ for large n.

4.4. Large-sample variance of $T^{(5)}$

We begin by defining $W_{42} = \frac{1}{3}(X_1 + X_2 + X_3) - X_4$, $W_{43} = \frac{1}{3}(X_2 + X_3 + X_4) - X_1$, so that $W_{42}|X_1 \sim N(\frac{1}{3}(X_1 - \mu), \frac{11}{9}\sigma^2)$ and $W_{43}|X_1 \sim N(-(X_1 - \mu), \frac{1}{3}\sigma^2)$. Let Z and z be as before in (4.19)-(4.20). We can express

$$\begin{aligned} I_{42} &= E[|\tfrac{1}{3}(x_1 + X_2 + X_3) - X_4|] \\ &\quad + E[|\tfrac{1}{3}(x_1 + X_2 + X_4) - X_3|] \\ &\quad + E[|\tfrac{1}{3}(x_1 + X_3 + X_4) - X_2|] \\ &= \sigma\left[\sqrt{\frac{22}{\pi}}\exp\left(-\frac{z^2}{22}\right) + 2z\Phi\left(\frac{z}{\sqrt{11}}\right) - z\right], \end{aligned} \tag{4.22}$$

and

$$\begin{aligned} I_{43} &= E[|\tfrac{1}{3}(X_2 + X_3 + X_4) - x_1|] \\ &= \sigma\left[\sqrt{\frac{2}{3\pi}}\exp\left(-\frac{3z^2}{2}\right) + 2z\Phi(\sqrt{3}z) - z\right]. \end{aligned} \tag{4.23}$$

Hence, we obtain

$$\zeta_1^{(5)} = \frac{3\pi}{128}E[I_{42} + I_{43}]^2 - \sigma^2, \tag{4.24}$$

but we can evaluate the expectation of each requisite term in (4.24) as follows:

$$E[I_{42}^2] = \sigma^2 E\left[\frac{22}{\pi}\exp\left(-\frac{Z^2}{11}\right) + 4Z^2\Phi^2\left(\frac{Z}{\sqrt{11}}\right) - 2\sqrt{\frac{22}{\pi}}Z\exp\left(-\frac{Z^2}{22}\right)\right.$$

$$\left. - 4Z^2\Phi\left(\frac{Z}{\sqrt{11}}\right) + 4\sqrt{\frac{22}{\pi}}Z\exp\left(-\frac{Z^2}{22}\right)\Phi\left(\frac{Z}{\sqrt{11}}\right) + Z^2\right]$$

$$= \sigma^2\left[\frac{22\sqrt{11}}{\pi\sqrt{13}} + \frac{2}{\pi}\sin^{-1}\left(\frac{1}{12}\right) + \frac{\sqrt{11}}{3\pi\sqrt{13}} + \frac{11\sqrt{11}}{3\pi\sqrt{13}}\right]$$

$$= \sigma^2\left[\frac{26\sqrt{11}}{\pi\sqrt{13}} + \frac{2}{\pi}\sin^{-1}\left(\frac{1}{12}\right)\right]; \tag{4.25}$$

$$E[I_{43}^2] = \sigma^2 E\left[\frac{2}{3\pi}\exp(-3Z^2) + 4Z^2\Phi^2(\sqrt{3}Z) - 2\sqrt{\frac{2}{3\pi}}Z\exp\left(-\frac{3Z^2}{2}\right)\right.$$

$$\left. - 4Z^2\Phi(\sqrt{3}Z) + 4\sqrt{\frac{2}{3\pi}}Z\exp\left(-\frac{3Z^2}{2}\right)\Phi(\sqrt{3}Z) + Z^2\right]$$

$$= \sigma^2\left[\frac{2}{3\pi\sqrt{7}} + \frac{2}{\pi}\sin^{-1}\left(\frac{3}{4}\right) + \frac{9}{3\pi\sqrt{7}} + \frac{3}{3\pi\sqrt{7}}\right]$$

$$= \sigma^2\left[\frac{14}{3\pi\sqrt{7}} + \frac{2}{\pi}\sin^{-1}\left(\frac{3}{4}\right)\right]; \tag{4.26}$$

and

$$E[I_{42}I_{43}] = \sigma^2 E\left[\frac{2\sqrt{11}}{\pi\sqrt{3}}\exp\left(-\frac{17Z^2}{11}\right) + \frac{2\sqrt{2}}{\sqrt{3\pi}}Z\Phi\left(\frac{Z}{\sqrt{11}}\right)\exp\left(-\frac{3Z^2}{2}\right)\right.$$

$$-\sqrt{\frac{2}{3\pi}}Z\exp\left(-\frac{3Z^2}{2}\right) + 2\sqrt{\frac{22}{\pi}}Z\exp\left(-\frac{Z^2}{22}\right)\Phi(\sqrt{3}Z) + Z^2$$

$$+ 4Z^2\Phi\left(\frac{Z}{\sqrt{11}}\right)\Phi(\sqrt{3}Z) - 2Z^2\Phi(\sqrt{3}Z) - \sqrt{\frac{22}{\pi}}Z\exp\left(-\frac{Z^2}{22}\right)$$

$$- 2Z^2\Phi\left(\frac{Z}{\sqrt{11}}\right)\Bigg]$$

$$= \sigma^2\left[\frac{22\sqrt{3}}{3\pi\sqrt{45}} + \frac{\sqrt{3}}{6\pi\sqrt{45}} + \frac{2}{\pi}\sin^{-1}\left(\frac{1}{4}\right) + \frac{121\sqrt{3}}{6\pi\sqrt{45}} + \frac{14\sqrt{3}}{6\pi\sqrt{45}}\right]$$

$$= \sigma^2\left[\frac{10\sqrt{3}}{\pi\sqrt{5}} + \frac{2}{\pi}\sin^{-1}\left(\frac{1}{4}\right)\right]. \tag{4.27}$$

Recall that evaluations of terms such as

$$E\left(Z^2\Phi^2\left(Z / \sqrt{11}\right)\right),\ E\left(Z^2\Phi\left(Z / \sqrt{11}\right)\right), \text{ and } E\left(Z\exp(-Z^2 / 22)\Phi\left(Z / \sqrt{11}\right)\right)$$

involved in (4.25) as well as similar terms involved in (4.26)-(4.27) use Lemmas 4.6, 4.2, and 4.5 as needed.

Next, by combining (4.25)-(4.27), we can express (4.24) as

$$\zeta_1^{(5)} = \frac{3\pi}{128}E[I_{42}^2 + I_{43}^2 + 2I_{42}I_{43}] - \sigma^2$$

$$= \sigma^2\left[\frac{3}{128}\left(2\sqrt{143} + 2\sin^{-1}\left(\frac{1}{12}\right) + \frac{14}{3\sqrt{7}} + 2\sin^{-1}\left(\frac{3}{4}\right)\right.\right.$$

$$\left.\left. + 20\sqrt{\frac{3}{5}} + \sin^{-1}\left(\frac{1}{4}\right)\right) - 1\right]. \tag{4.28}$$

Hence, the asymptotic variance of $T^{(5)}$ is $\frac{16}{n}\zeta_1^{(5)} + O(n^{-1})$.

4.5. Some additional lemmas

In this subsection, we state and prove some additional results as lemmas. These lemmas are needed to fill in some of the details explained in Subsections 4.1-4.4. We write $\Phi(y), \phi(y)$, respectively, for the distribution function and the probability density function of Y, a standard normal random variable, $-\infty < y < \infty$.

Lemma 4.1. *For any two arbitrary positive real numbers c and d, we have*

$$E[\Phi(cY)\Phi(dY)] = \frac{1}{4} + \frac{1}{2\pi}\sin^{-1}\left(\frac{cd}{\sqrt{1+c^2}\sqrt{1+d^2}}\right). \tag{4.29}$$

Proof. Let U, V be i.i.d. $N(0, 1)$ random variables, both independent of Y. Then,

$$\begin{aligned}
&P\{(U \le cY) \cap (V \le dY)\} \\
&\quad = \int_{\Re} P((U \le cY) \cap (V \le dY) | Y = y)\phi(y)dy \\
&\quad = \int_{\Re} P((U \le cy) \cap (V \le dy))\phi(y)dy \\
&\quad = \int_{\Re} P(U \le cy)P(V \le dy)\phi(y)dy \\
&\quad = \int_{\Re} \Phi(cY)\Phi(dY)\phi(y)dy \\
&\quad = E[\Phi(cY)\Phi(dY)].
\end{aligned} \tag{4.30}$$

Now, let us denote $W = U - cY$, $X = V - dY$, $R = \frac{W}{\sqrt{1+c^2}}$, and $S = \frac{X}{\sqrt{1+d^2}}$. Then, $\begin{pmatrix} W \\ X \end{pmatrix} \sim N_2(0, 0, (1+c^2), (1+d^2), \rho)$, where $\rho = \frac{cd}{\sqrt{1+c^2}\sqrt{1+d^2}}$. Then, from (4.30), we have

$$\begin{aligned}
E[\Phi(cY)\Phi(dY)] &= P\{(U - cY \le 0) \cap (V - dY \le 0)\} \\
&= P\{(W \le 0) \cap (X \le 0)\} \\
&= P\{(R \le 0) \cap (S \le 0)\} \\
&= \frac{1}{2} P\{\frac{R}{S} \geqslant 0\}.
\end{aligned} \tag{4.31}$$

Let us define $T = \frac{R}{S}$, which has its probability density function given by

$$g(t) = \frac{\sqrt{1-\rho^2}}{\pi\{(t-\rho)^2 + (1-\rho^2)\}}, \; t \in \Re,$$

so that we have

$$E[\Phi(cY)\Phi(dY)] = \frac{1}{2} P(T \geqslant 0) = \frac{1}{2} \int_{\Re+} g(t)dt = \frac{1}{2\pi} \int_{-\frac{\rho}{\sqrt{1-\rho^2}}}^{\infty} \frac{ds}{1+s^2}$$

$$= \frac{1}{2\pi} [\tan^{-1}(s)]_{s=-\frac{\rho}{\sqrt{1-\rho^2}}}^{s=\infty} = \frac{1}{2\pi}\left[\frac{\pi}{2} - \tan^{-1}\left(-\frac{\rho}{\sqrt{1-\rho^2}}\right)\right]$$

$$= \frac{1}{4} + \frac{1}{2\pi} \sin^{-1}(\rho), \tag{4.32}$$

which completes the proof. □

Lemma 4.2. *For any arbitrary positive real number c, we have*

$$(Y^2\Phi(cY)) = \frac{1}{2}.$$

Proof. We observe that

$$E[Y^2\Phi(cY)] = \int_{\Re} y^2\Phi(cy)\phi(y)dy$$

$$= \int_0^{\infty} u^2[1-\Phi(cu)]\phi(u)du + \int_0^{\infty} y^2\Phi(cy)\phi(y)dy$$

$$= \int_0^{\infty} u^2\phi(u)du, \tag{4.33}$$

which reduces to $\frac{1}{2}$. □

Lemma 4.3. *Suppose that a random variable W has the student's t distribution with one degree of freedom. Then, for any fixed $c > 0$, we have*

$$P(W \leq c) = 2\int_0^\infty \Phi(cy)\phi(y)dy.$$

Proof. We begin with independent random variables U, V, where $U \sim N(0, 1)$ and $V \sim \chi_1^2$, so that we may denote $W = \frac{U}{\sqrt{V}}$. Now, we obtain

$$P(W \leq c) = P\left(U \leq c\sqrt{V}\right) = \frac{1}{\sqrt{2\pi}\sqrt{v}}\int_0^\infty \Phi(cv)\exp\left(-\frac{v}{2}\right)dv$$

$$= 2\int_0^\infty \Phi(cy)\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{y^2}{2}\right)dy = 2\int_0^\infty \Phi(cy)\phi(y)dy,$$

which proves the result. $\square$

Lemma 4.4. *Suppose that a pair of random variables (U, V) has the bivariate normal distribution, $N_2(0, 0, 1, 1, -\frac{1}{2})$. Then, we have*

$$E[|UV|] = \frac{\sqrt{3}}{\pi} + \frac{1}{\pi}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right).$$

Proof. We first observe that

$$E(|UV|) = E[|V|E\{|U|\,|V\}]$$

$$= E\left[|V|\left\{\frac{\sqrt{3}}{2}\cdot\sqrt{\frac{2}{\pi}}\exp\left(-\frac{V^2}{6}\right) - \frac{V}{2}\left(1 - 2\Phi\left(\frac{V}{\sqrt{3}}\right)\right)\right\}\right]$$

$$= \sqrt{\frac{3}{2\pi}}E\left[|V|\exp\left(-\frac{V^2}{6}\right)\right] - E\left[\frac{1}{2}V|V|\right] + E\left[V|V|\Phi\left(\frac{V}{\sqrt{3}}\right)\right]. \quad (4.34)$$

Clearly, we have

$$E\left[|V| \exp\left(-\frac{V^2}{6}\right)\right] = \frac{2}{\sqrt{2\pi}} \int_0^\infty v \exp\left(-\frac{2}{3}v^2\right) dv = \frac{3}{2\sqrt{2\pi}},$$

and $E[V|V|] = 0.$ (4.35)

Next, we handle the remaining term from (4.34) and write with $c > 0$

$$\begin{aligned}
E[V|V|\Phi(cV)] &= \int_{-\infty}^{0} v|v|\Phi(cv)\phi(v)dv + \int_0^\infty v|v|\Phi(cv)\phi(v)dv \\
&= -\int_0^\infty v^2\Phi(-cv)\phi(v)dv + \int_0^\infty v^2\Phi(cv)\phi(v)dv \\
&= -\int_0^\infty v^2\phi(v)dv + 2\int_0^\infty v^2\Phi(cv)\phi(v)dv \\
&= -\frac{1}{2} + 2\left[\int_0^\infty \Phi(cv)\left\{\frac{d^2}{dv^2}\phi(v) + \phi(v)\right\}dv\right] \\
&= -\frac{1}{2} + 2\int_0^\infty \Phi(cv)\phi(v)dv + 2c\int_0^\infty v\,\phi(cv)\phi(v)dv \\
&= -\frac{1}{2} + P(W \le c) + \frac{c}{\pi(1+c^2)}, \qquad (4.36)
\end{aligned}$$

where $W \sim t_1$, using Lemma 4.3. That is,

$$E\left[V|V|\Phi\left(\frac{V}{\sqrt{3}}\right)\right] = \frac{1}{\pi}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) + \frac{\sqrt{3}}{4\pi}.$$

Now, combining the previous steps (4.34)-(4.36), we complete the proof. □

Lemma 4.5. *For any two arbitrary positive real numbers c and d, we have*

$$E\left[Y\Phi(cY)\exp\left(-\frac{d^2Y^2}{2}\right)\right] = \frac{c}{\sqrt{2\pi}(1+d^2)\sqrt{1+c^2+d^2}}.$$

Proof. First, we express $E\left[Y\Phi(cY)\exp\left(-\frac{d^2Y^2}{2}\right)\right]$ as follows:

$$\int_{-\infty}^{\infty}\frac{y}{\sqrt{2\pi}}\Phi(cy)\exp\left(-\frac{d^2}{2}y^2\right)\exp\left(-\frac{1}{2}y^2\right)dy$$

$$= \frac{1}{(1+d^2)}\int_{-\infty}^{\infty} y\Phi\left(\frac{c}{\sqrt{1+d^2}}y\right)\phi(y)dy$$

$$= \frac{1}{(1+d^2)}\left[2\int_0^{\infty} y\Phi\left(\frac{c}{\sqrt{1+d^2}}y\right)\phi(y)dy - \int_0^{\infty} y\phi(y)dy\right]$$

$$= \frac{1}{(1+d^2)}\left[-2\int_0^{\infty}\Phi\left(\frac{c}{\sqrt{1+d^2}}y\right)\left(\frac{d}{dy}(\phi(y))\right)dy - \int_0^{\infty} y\phi(y)dy\right]$$

$$= \frac{1}{(1+d^2)}\left[-2\left\{-\frac{1}{2\sqrt{2\pi}} - \frac{c}{2\sqrt{2\pi}\sqrt{1+c^2+d^2}}\right\} - \frac{1}{\sqrt{2\pi}}\right]$$

$$= \frac{c}{(1+d^2)\sqrt{2\pi}\sqrt{1+c^2+d^2}}, \tag{4.37}$$

since we have

$$\int_0^{\infty}\Phi\left(\frac{c}{\sqrt{1+d^2}}y\right)\frac{d}{dy}\phi(y)dy$$

$$= \Phi\left(\frac{c}{\sqrt{1+d^2}}y\right)\phi(y)\Bigg]_{y=0}^{y=\infty} - \frac{c}{\sqrt{1+d^2}}\int_0^{\infty}\phi\left(\frac{c}{\sqrt{1+d^2}}y\right)\phi(y)dy$$

$$= -\frac{1}{2\sqrt{2\pi}} - \frac{c}{2\sqrt{2\pi}\sqrt{1 + c^2 + d^2}}. \tag{4.38}$$

Now, the proof is complete. □

Lemma 4.6. *For any two arbitrary positive real numbers c and d, we have*

$$E[Y^2\Phi(cY)\Phi(dY)] = \frac{1}{4} + \frac{1}{2\pi}\sin^{-1}\left(\frac{cd}{\sqrt{1 + c^2}\sqrt{1 + d^2}}\right)$$

$$+ \frac{cd}{2\pi(1 + c^2)\sqrt{1 + c^2 + d^2}}$$

$$+ \frac{cd}{2\pi(1 + d^2)\sqrt{1 + c^2 + d^2}}.$$

Proof. Again, we first express $E[Y^2\Phi(cY)\Phi(dY)]$ as follows:

$$\int_{-\infty}^{\infty} y^2\Phi(cY)\Phi(dY)\phi(y)dy$$

$$= \int_{-\infty}^{\infty} \Phi(cY)\Phi(dY)\phi(y)dy + \int_{-\infty}^{\infty} \Phi(cY)\Phi(dY)\frac{d^2\phi(y)}{dy^2}dy$$

$$= E[\Phi(cY)\Phi(dY)] - \int_0^{\infty} \Phi(cY)\Phi(dY)\frac{d}{dy}(y\phi(y))dy$$

$$- \int_0^{\infty} \Phi(-cY)\Phi(-dY)\frac{d}{dy}(y\phi(y))dy$$

$$= E[\Phi(cY)\Phi(dY)] - 2\int_0^{\infty} \Phi(cY)\Phi(dY)\frac{d}{dy}(y\phi(y))dy - \int_0^{\infty} \frac{d}{dy}(y\phi(y))dy$$

$$+\int_0^\infty \Phi(cy)\frac{d}{dy}(y\phi(y))dy + \int_0^\infty \Phi(dy)\frac{d}{dy}(y\phi(y))dy$$

$$= E[\Phi(cY)\Phi(dY)] - 2I_1 - I_2 + I_3 + I_4, \text{ say.} \tag{4.39}$$

Evaluation of the term $E[\Phi(cY)\Phi(dY)]$ seen in the last step of (4.39) will require Lemma 4.1. Next, we proceed to evaluate I_1 through I_4.

$$I_1 = \int_0^\infty \Phi(cY)\Phi(dY)\frac{d}{dy}(y\phi(y))dy$$

$$= \Phi(cY)\Phi(dY)y\phi(y)]_0^\infty - c\int_0^\infty y\Phi(dy)\phi(cy)\phi(y)dy$$

$$- d\int_0^\infty y\Phi(cy)\phi(dy)\phi(y)dy$$

$$= I_{11} + I_{12}. \tag{4.40}$$

The first term in (4.40) obviously reduces to zero. The term I_{11}, and similarly I_{12}, is simplified as

$$I_{11} = -c\int_0^\infty y\Phi(dy)\phi(cy)\phi(y)dy$$

$$= -\frac{c^2}{\sqrt{2\pi}}\int_0^\infty y\Phi(dy)\times\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{y^2}{2}(1+c^2)\right)dy$$

$$= -\frac{c^2}{\sqrt{2\pi}(1+c^2)}\int_0^\infty y\Phi\left(\frac{dy}{\sqrt{1+c^2}}\right)\phi(y)dy$$

$$= \frac{c^2}{\sqrt{2\pi}(1+c^2)}\int_0^\infty \Phi\left(\frac{dy}{\sqrt{1+c^2}}\right)\frac{d}{dy}(\phi(y))dy$$

$$= \frac{c^2}{\sqrt{2\pi}(1+c^2)}\left[\left(\Phi\left(\frac{dy}{\sqrt{1+c^2}}\right)\phi(y)\right)_0^\infty\right.$$

$$\left. - \frac{d}{\sqrt{1+c^2}}\int_0^\infty \phi\left(\frac{dy}{\sqrt{1+c^2}}\right)\frac{d}{dy}(\phi(y))dy\right]$$

$$= -\frac{c}{4\pi(1+c^2)} - \frac{cd}{4\pi(1+c^2)\sqrt{1+c^2+d^2}}. \tag{4.41}$$

Also, we have

$$I_2 = \int_0^\infty \frac{d}{dy}(y\phi(y))dy = \int_0^\infty \phi(y)dy - \int_0^\infty y^2\phi(y)dy = 0;$$

$$I_3 = \int_0^\infty \Phi(cy)\frac{d}{dy}(y\phi(y))dy = (y\Phi(cy)\phi(y))_0^\infty - c\int_0^\infty y\phi(cy)\phi(y)dy$$

$$= -\frac{c}{2\pi}\int_0^\infty y\exp\left(-\frac{y^2(1+c^2)}{2}\right)dy = -\frac{c}{2\pi(1+c^2)};$$

$$I_4 = \int_0^\infty \Phi(dy)\frac{d}{dy}(y\phi(y))dy = -\frac{d}{2\pi(1+d^2)}. \tag{4.42}$$

Combining (4.39)-(4.42), we get the desired result. □

5. Some Concluding Comments

The existing unbiased estimators $T^{(1)}$ and $T^{(2)}$ for σ are well-known in the literature. Both depended upon U-statistics with symmetric kernels of degree two. The new proposed unbiased estimators $T^{(3)}$, $T^{(4)}$, and $T^{(5)}$ for σ depend upon U-statistics with symmetric kernels of degree three, four, and four, respectively. From this investigation, it becomes clear that $T^{(3)}$, $T^{(4)}$, and $T^{(5)}$ for σ; (i) go practically head-to-head with $T^{(1)}$ and $T^{(2)}$, (ii) $T^{(4)}$ beats $T^{(2)}$, and (iii) $T^{(3)}$ very nearly beats $T^{(2)}$, whether n is small or moderately large.

Clearly, $T^{(1)}$ is more efficient than either $T^{(2)}$ or $T^{(3)}$ whatever is the sample size. That is to be expected since $T^{(1)}$ is the MVUE for σ. But, if we use $T^{(3)}$ in place of $T^{(2)}$ (when $n > 55$) or instead of $T^{(1)}$ (whatever be n) for estimating σ, then one will obviously encounter some loss of information, but the loss of efficiency is very small. In other words, the kinds of constructions proposed here may be advantageous and fruitful in data analyses.

In a recent unpublished technical report, Mukhopadhyay and Chattopadhyay [18] utilized analogous constructions of unbiased estimators of $\sigma^2 \equiv \sigma^2(F)$, in the case of an arbitrary distribution function $F(.)$. In the distribution-free situation, all such unbiased estimators ultimately came down to coincide with S^2. This was an interesting and striking result in its own right leading to new interpretations of a sample variance. While that unpublished technical report has no direct or indirect bearing on our present investigation, we have cited it here for completeness only.

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